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MASS AND BINDING FROM DIRECT INTERACTION †

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Abstract An exactly soluble model field theory is proposed to describe physical particles on a unified basis By starting from a direct interaction of Fermi type for particles with vanishing bare mass, the simultaneous increase of both physical mass and binding effects, caused by the interaction, is made explicit Convergence of the local theory is assured by a resonance effect The exact solution for the compound-particle is confronted with those obtained from various approximation schemes, viz one-time and many-time Tamm-Dancoff and Bethe-Salpeter equations The coupling of the compound-particle to its constituents is determined

1. Introduction

The self-interaction of a single field corresponding to bare particles with vanishing mass should furnish these particles with non-vanishing physical mass and at the same time give rise to binding effects between the massive particles, thus leading to the formation of compound-particles While a rigorous formulation of this idea underlying several unified elementary particle theories [1]) is still missing, the fair agreement of some of the calculated compound-particle masses with experimental data points to the desirability by starting from massless particles, of making explicit the simultaneous increase of both particles mass and binding effects and of analysing the meaning and efficiency of various approximation schemes for the description of these effects In a rigorous way these problems can only be discussed at present in terms of models which are simple enough to be soluble exactly but still possess part of the general properties and structures the realistic theories should contain A model field theory of this type is proposed in the following

In a relativistic theory of a single field $A(x, t)$, describing particles labelled A, coupled with itself through a direct Fermi interaction A^+AA^+A, this interaction can

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be decomposed schematically into a binding term $a^+ a^+ aa$ describing the binding or scattering of two A particles (fig 1) a dressing term $a^+a^+a^+a+a^+aaa$ which gives rise to self-mass (fig 2) and a vacuum term $a^+ a^+ a^+ a^+ + aaaa$ responsible for vacuum fluctuation processes Here, a^+ and a are the creation and annihilation operators for a bare A particle. In what follows we shall not consider the vacuum effects In spite of this simplification no exact treatment of the remaining terms is possible since one cannot account for a complete iteration of the dressing term † The essential simplification that makes the model exactly soluble consists of replacing the above dressing term by the expression $b^+ b^+ b^+ a + a^+ bbb$ (fig 3) Here b^+ and b are creation and annihilation operators of particles labelled B, described [2]) by a new auxiliary field $B(\boldsymbol{x}, t)$, which later will be eliminated Obviously, in virtue of the missing symmetry between creation and annihilation operators in the interaction the model becomes a non-relativistic one which, however, can be solved exactly

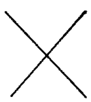

Fig 1 Binding and scattering graph for A particles

Fig 2 Dressing graph for A particles

Fig 3 Dressing of A particles by B particles

Suppose now the bare A particles have an arbitrarily small bare mass so that their bare rest-energy – in contrast with their kinetic energy – can be put equal to zero The effect of the B particles is now just to dress the bare A particles, thereby furnishing the latter with a positive, finite physical rest-energy, or physical mass (sect 2) In a non-relativistic theory there is no reason to identify inertial mass m with rest-mass m_0 in the energy function $\varepsilon(\boldsymbol{k}) = (\boldsymbol{k}^2/2m)+m_0$ of the particle By dressing or mass formation we mean, then, the increase of the rest-mass of the particle from $m_0 = 0$ to $m_0 > 0$, the inertial mass remaining unaltered The B field is now to be chosen in such a way that this dressing process is the only effect the B particles give rise to. The B particles, then, do not manifest themselves otherwise, that is to say, there will be no scattering or binding of A particles through intermediate B particles, and the B particles are finally eliminated by making their bare mass infinitely large

The model thus obtained turns out to be local and finite We shall show in fact how a field theory with Fermi interaction can be made finite in the local limit through a resonance effect without the necessity of introducing ghost states The bound state (meson) formed by two massive A particles (nucleons) will be calculated exactly and will be shown to be caused entirely by the primary self-interaction of the A particles, i e , by the term $a^+ a^+ aa$, without participation of intermediate B particles which only turn the neutrino-like bare A particles into massive ones (sect 3)

In sect. 4 the exact solutions of the model are confronted with those obtained by various approximation schemes Perturbation theory gives the exact result for the

† An attempt in this direction has been made by one of us [10])

single-particle problem in spite of the singular structure S_F^3 of the self-energy of the A particle The bound state problem is treated in the one-time and many-time Tamm-Dancoff approximation The absence of any relative-time correlation between the two constituents of the bound state in the $n = 2$ one-time approximation prevents the appearance of virtual clouds around both of them so that we are left with a bound state of two massless A particles of negative energy On the other hand, the $n = 2$ two-time approximation gives a positive-energy solution, but not the exact one, however, since the relative-time correlation in this case accounts for the dressing of only one of the constituents, the other remaining massless If we proceed to the $n = 4$ approximation, this missing energy will be supplied, thus leading to the exact solution. The Bethe-Salpeter equation corresponding to the ladder diagram likewise gives the exact result since here the relative-time correlation enters in a symmetrical way so that both particles remain dressed It is obvious from this, that the one-time Tamm-Dancoff method is generally less efficient than the other methods.

In sect. 5 the coupling of the compound particle to its constituents is determined from the bound state propagator which is essentially given by the inverse of the Bethe-Salpeter equation In a forthcoming paper we shall demonstrate that the Fermi theory can be considered as the limiting case of a Yukawa theory with vanishing Z_3 renormalization Although being non-relativistic, the model may be viewed as a truncated form of a relativistic theory

2. The One-Particle Problem

The system to be considered is defined by the Hamiltonian

$$H = H_0 + H_1, \tag{1}$$

with

$$H_0 = \int \mathrm{d}\boldsymbol{x} \left[\frac{1}{2m} \nabla A^+ \nabla A - MB^+B\right], \tag{2}$$

$$H_1 = (2\pi)^3 \lambda_1 \int \mathrm{d}\boldsymbol{x}\, A^+A^+AA + (2\pi)^3 \lambda_2 \int \mathrm{d}x [B^+B^+B^+A + A^+BBB], \tag{3}$$

where m and M are the kinetic and bare masses of A and B particles, respectively The rest-energy of the bare A particle (a term $-mA^+A$) has been put equal to zero in accordance with what has just been said and we have neglected the kinetic part $(1/2M)\nabla B^+ \nabla B$ of the B parricle by requiring $M \to \infty$ The parameters $\lambda_1 < 0$ and λ_2 are real coupling constants and we include tacitly two auxiliary cutoffs K_1 and K_2 in the terms A^+A^+AA and $B^+B^+B^+A + A^+BBB$, respectively, making finally $K_1 \to \infty$, $K_2 \to \infty$ We set $\hbar = c = 1$

Both A and B particles are conveniently quantized according to Bose statistics For A we have

$$[A(\boldsymbol{x}, t), A^+(\boldsymbol{x}', t)] = \delta(\boldsymbol{x} - \boldsymbol{x}'), \qquad [a(\boldsymbol{k}), a^+(\boldsymbol{k}')] = \delta(\boldsymbol{k} - \boldsymbol{k}'), \tag{4}$$

while for B we take an indefinite metric which, however, has no relation with the convergence of the theory but is due to the non-relativistic character of the model. In a theory in which physical and bare vacuum coincide, the physical energy of a particle is always smaller than or equal to the bare energy if quantization is performed with a positive definite metric Indeed, putting $a^+(\boldsymbol{k})|0\rangle = c|1, \boldsymbol{k}\rangle + \sum c_i\phi_i$, where the ϕ_i represent scattering states, whose energies E_i, for stability reasons, must be larger than the energy of the state of one particle, we find

$$\langle 0|aHa^+|0\rangle = \langle 0|aH_0a^+|0\rangle + \langle 0|aH_1a^+|0\rangle = \langle 0|aH_0a^+|0\rangle = E_{\text{bare}},$$

where use has been made of the fact that $\langle 0|aH_1a^+|0\rangle = 0$, if the interaction is written in terms of normal products On the other hand, expanding in terms of eigenstates of the total Hamiltonian, we obtain $\langle 0|aHa^+|0\rangle = c^2E + \sum c^2 E_i$ Subtracting both equalities gives $c^2(E-E_{\text{bare}}) + \sum c_i^2(E_i - E_{\text{bare}}) = 0$, which implies $E \leqq E_{\text{bare}}$ This implies a quantization with indefinite metric if we insist on having a positive physical mass for the A particle We observe that the theory is invariant under the transformation $A \to Ae^{i\gamma}$, $B \to Be^{\frac{1}{3}i\gamma}$, which entails that

$$n = n_a + \tfrac{1}{3}n_b = \int \mathrm{d}\boldsymbol{k}\, a^+(\boldsymbol{k})a(\boldsymbol{k}) + \tfrac{1}{3}\int \mathrm{d}\boldsymbol{k}\, b^+(\boldsymbol{k})b(\boldsymbol{k}) \tag{5}$$

is a good quantum number and we have separate sectors

The commutation relations for B particles now read

$$[B(\boldsymbol{x}, t)B^+(\boldsymbol{x}', t)] = -\delta(\boldsymbol{x}-\boldsymbol{x}'), \qquad [b(\boldsymbol{k}), b^+(\boldsymbol{k})] = -\delta(\boldsymbol{k}-\boldsymbol{k}') \tag{6}$$

In momentum space, the Hamiltonian is given by

$$\begin{aligned} H = &\int \mathrm{d}\boldsymbol{k} \left[\frac{k^2}{2m}\, a^+(\boldsymbol{k})a(\boldsymbol{k}, - Mb^+(\boldsymbol{k})b(\boldsymbol{k})\right] \\ &+\lambda_1 \int \mathrm{d}\boldsymbol{k}_1 \quad \mathrm{d}\boldsymbol{k}_4\, a^+(\boldsymbol{k}_1)a^+(\boldsymbol{k}_2)a^+(\boldsymbol{k}_3)a^+(\boldsymbol{k}_4)\delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4) \\ &+\lambda_2 \int \mathrm{d}\boldsymbol{k}_1 \quad \mathrm{d}\boldsymbol{k}_4[b^+(\boldsymbol{k}_1)b^+(\boldsymbol{k}_2)b^+(\boldsymbol{k}_3)a(\boldsymbol{k}_4) + \text{h c}]\delta(\boldsymbol{k}_4-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3), \end{aligned} \tag{7}$$

where $\boldsymbol{k}_i^2 \leqq K_1^2$ in the term with λ_1 and $\boldsymbol{k}_i^2 \leqq K_2^2$ in the term with λ_2 and $K_i \to \infty$ at the end of all calculations

Let $|0\rangle$ be the bare vacuum which here coincides with the physical one $H|0\rangle = 0$. The state of a physical A particle with momentum $\boldsymbol{k}$ can be represented by

$$\begin{aligned} |a, \boldsymbol{k}\rangle = &\, a^+(\boldsymbol{k})|0\rangle \\ &+ \int \mathrm{d}\boldsymbol{k}_1 \quad \mathrm{d}\boldsymbol{k}_3\, f_0(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)\delta(\boldsymbol{k} - \sum_1^3 \boldsymbol{k}_i)b^+(\boldsymbol{k}_1)b^+(\boldsymbol{k}_2)b^+(\boldsymbol{k}_3)|0\rangle. \end{aligned} \tag{8}$$

In front of $|a, \boldsymbol{k}\rangle$ we have dropped a normalization factor N_0, which, as is easily seen, tends to unity if, as we require, M tends to infinity, in which case no scattering of B particles occurs, since then $E < 3M$. Inserting (8) into the eigenvalue equation

$$H|a, \boldsymbol{k}\rangle = E|a, \boldsymbol{k}\rangle,$$

we obtain the two equations

$$(E-3M)f_0(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = \lambda_2 \prod_1^3 \Theta(K_2^2-\boldsymbol{k}_i^2),$$

$$E-\frac{1}{2m}\boldsymbol{k}^2 = -3!\lambda_2 \int \mathrm{d}\boldsymbol{k}_1 \quad \mathrm{d}\boldsymbol{k}_3 f_0(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)\delta(\boldsymbol{k}-\sum_1^3 \boldsymbol{k}_i) \prod_1^3 \Theta(K_2^2-\boldsymbol{k}_i^2), \tag{9}$$

where the cutoff has explicitly been introduced through the step function Θ. From eq. (9) it follows that for $E < 3M$,

$$E-\boldsymbol{k}^2/2m = 3!\lambda_2^2 G(\boldsymbol{k})/(3M-E), \tag{10}$$

where

$$G(\boldsymbol{k}) = \int \mathrm{d}\boldsymbol{k}_1 \quad \mathrm{d}\boldsymbol{k}_3 \delta(\boldsymbol{k}-\sum_1^3 \boldsymbol{k}_i) \prod_1^3 \Theta(K_2^2-\boldsymbol{k}_i^2) = \tfrac{5}{6}K_2^6\pi^2+o(K_2^2, \boldsymbol{k}^2) \tag{11}$$

Hence,

$$E-\boldsymbol{k}^2/2m = [5K_2^6\lambda_2^2\pi^2-o(K_2^2, \boldsymbol{k}^2)]/(3M-E) \tag{12}$$

Let now $K_2 \to \infty$, $M \to \infty$ in such a way that

$$\lim_{K_2\to\infty,\, M\to\infty} [5K_2^6\pi^2/3M] = \alpha > 0, \tag{13}$$

where α is finite, then the energy of the physical A particle becomes

$$E = \frac{1}{2m}\boldsymbol{k}^2+m_0, \qquad m_0 = \alpha\lambda_2^2 > 0, \tag{14}$$

with the positive rest-energy m_0. The originally massless bare A particle gains this mass through its interaction with virtual B particles. We observe that the second

Fig. 4. Self-energy part of A particles.

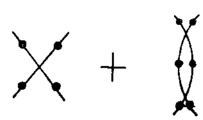

Fig. 5. Scattering or binding graph for dressed A particles.

solution [3]) of eq. (12) when $M < \infty$ (ghost) disappears in the limit $M \to \infty$. Obviously there is no scattering of 3 B particles (cf. the appendix). In the limit $M \to \infty$ we have $f \approx -\lambda_2/3M$. Hence, in this limit $|a, \boldsymbol{k}\rangle$ is a correctly normalized state, $\langle a, \boldsymbol{k}|a, \boldsymbol{k}'\rangle = \delta(\boldsymbol{k}-\boldsymbol{k}')[1-m_0 o(1/M)] \to \delta(\boldsymbol{k}-\boldsymbol{k}')$, and $|a, \boldsymbol{k}\rangle \to a^+(\boldsymbol{k})|0\rangle$ in the sense of strong convergence.

3. Bound-State Problem

For the bound state $|2a\rangle$ of two physical A particles, satisfying the eigenvalue equation

$$H|2a\rangle = E|2a\rangle,$$

we make in the centre-of-mass system the ansatz

$$\begin{aligned}|2a\rangle = &\int \mathrm{d}\boldsymbol{k}\, f(\boldsymbol{k}) a^+(\boldsymbol{k}) a^+(-\boldsymbol{k})|0\rangle \\ &+ \int \mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3\, g(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)\delta(\boldsymbol{k} - \sum_1^3 \boldsymbol{k}_i) \prod_1^3 b^+(\boldsymbol{k}_i) a^+(-\boldsymbol{k})|0\rangle \\ &+ \int \mathrm{d}\boldsymbol{k}_1 \ldots \mathrm{d}\boldsymbol{k}_6\, r(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_6)\delta(\sum_1^6 \boldsymbol{k}_i) b^+(\boldsymbol{k}_1) \ldots b^+(\boldsymbol{k}_6)|0\rangle, \end{aligned} \tag{15}$$

which gives rise to the following system of equations

$$(E - \boldsymbol{k}^2/m) f(\boldsymbol{k}) = 2\lambda_1 \int \mathrm{d}\boldsymbol{k}\, f(\boldsymbol{k}) - 3!\lambda_2 \int \prod_1^3 \mathrm{d}\boldsymbol{k}_i\, g(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)\delta(\boldsymbol{k} - \sum_1^3 \boldsymbol{k}_i), \tag{16a}$$

$$\begin{aligned}&[(E - 3M - \boldsymbol{k}^2/2m) g(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) - 2\lambda_2 f(\boldsymbol{k})]\delta(\boldsymbol{k} - \sum_1^3 \boldsymbol{k}_i) \\ &\qquad = -5!\lambda_2 \int \mathrm{d}\boldsymbol{k}_4\,\mathrm{d}\boldsymbol{k}_5\,\mathrm{d}\boldsymbol{k}_6\, r(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_6)\delta(\sum_1^6 \boldsymbol{k}_i)\delta(\boldsymbol{k} - \sum_1^3 \boldsymbol{k}_i),\end{aligned} \tag{16b}$$

$$(E - 3!M) r(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_6)\delta(\sum_1^6 \boldsymbol{k}_i) = \lambda_2 [g(\sum_1^3 \boldsymbol{k}_i, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)\delta(\sum_1^6 \boldsymbol{k}_i)]_s \tag{16c}$$

where the index s means symmetrization in all variables Substitution of (16c) into (16b) gives

$$\begin{aligned}&(E - (\sum_1^3 \boldsymbol{k}_i)^2/2m - 3M) g(\sum_1^3 \boldsymbol{k}_i, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) - 2\lambda_2 f(\sum_1^3 \boldsymbol{k}_i) \\ &\qquad = -5![\lambda_2^2/(E - 3!M)] \int \prod_4^6 \mathrm{d}\boldsymbol{k}_i [g(\sum_1^3 \boldsymbol{k}_i, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)\delta(\sum_1^6 \boldsymbol{k}_i)]_s .\end{aligned}$$

Taking now the limits $K_2 \to \infty$, $M \to \infty$ we arrive at

$$g(\sum_1^3 \boldsymbol{k}_i, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = -2\lambda_2 f(\sum_1^3 \boldsymbol{k}_i)/3M) + \lambda_2^2 S K_2^6/18M + o(M^{-1}), \tag{17}$$

where

$$S = -5\pi^2 g(\sum_1^3 \boldsymbol{k}_i, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) - K_2^{-6} \int \prod_4^6 \mathrm{d}\boldsymbol{k}_i [g(\sum_2^4 \boldsymbol{k}_i, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4)\delta(\sum_1^6 \boldsymbol{k}_i) + \ldots],$$

and thus

$$|S| \lessapprox 5^{\prime}|g(\sum_{1}^{3} \boldsymbol{k}_i, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)|\tfrac{5}{6}\pi^2$$

Then, from eqs. (17) and (13) we have

$$g(\sum_{1}^{3} \boldsymbol{k}_i, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = -2\lambda_2^2 f(\sum_{1}^{3} \boldsymbol{k}_i)/3M + o(1/M), \tag{18}$$

and substituting this into eq (15) we find the eigenvalue equation

$$(E-\boldsymbol{k}^2/m)f(\boldsymbol{k}) = (4\lambda_2^2/M)G(\boldsymbol{k})f(\boldsymbol{k}) + 2\lambda_1 \int \mathrm{d}\boldsymbol{k}' f(\boldsymbol{k}'),$$

which in virtue of eqs. (11) and (13) simplifies to

$$(E-(\boldsymbol{k}^2/m)-2m_0)f(\boldsymbol{k}) = 2\lambda_1 \int \mathrm{d}\boldsymbol{k}' f(\boldsymbol{k}') \tag{19}$$

From this equation we find for the energy E of the bound state the expression

$$\int_0^{K_1} \mathrm{d}|\boldsymbol{k}|/\boldsymbol{k}^2(E-\boldsymbol{k}^2/m-2m_0)^{-1} = 1/8\pi\lambda_1, \tag{20}$$

or

$$m^{\frac{3}{2}}(2m_0-E)^{\frac{1}{2}} \operatorname{arctg}[K_1/(2mm_0-Em_0)^{\frac{1}{2}}] = mK_1 + 1/8\pi\lambda_1 \tag{21}$$

Let us now perform the limiting processes $K_1 \to \infty$ and $\lambda_1 \to 0$ in such a way that

$$\lim_{K_1\to\infty,\, \lambda_1\to 0} [mK_1 + 1/8\pi\lambda_1] = \beta > 0, \tag{22}$$

where β is finite Then the energy of the compound-particle is just given by

$$E = 2m_0 - B, \tag{23}$$

with the binding energy

$$B = 4\beta^2/m^3\pi^2 > 0, \tag{24}$$

E is positive if β is chosen sufficiently small.

It is easily seen that the solutions

$$E = 2m_0 - B,$$

$$f(\boldsymbol{k}) = (E-\boldsymbol{k}^2/m-2m_0)^{-1}, \tag{25}$$

$$g(\sum_{1}^{3} \boldsymbol{k}_i, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = 2\lambda_1 f(\sum_{1}^{3} \boldsymbol{k}_i)/3M, \qquad g \to 0, \qquad r \to 0 \qquad \text{for } M \to \infty$$

are in fact the exact ones, the solution being unique if first the limit (13) and then the limit (22) are performed Eq (25) implies that $r \to 0$ and $g \to 0$ as $M \to \infty$ Hence, no B particles will participate in the bound state They merely furnish the massless A particles with rest-energy $m_0 > 0$ The binding forces are entirely caused by the primary self-interaction of the A particles This self-interaction $\lambda_1 \int d\boldsymbol{x} A^+ A^+ AA$ of the A particles is different from zero in spite of the limiting procedure (22), $\lambda_1 \to 0$, since at the same time the cutoff K_1 involved in $\int d\boldsymbol{x} A^+ A^+ AA$ tends to infinity so that, roughly speaking, the divergence of the product $A^+ A^+ AA$ of field operators is compensated by the vanishing of the coupling constant λ_1

A simple physical argument shows the necessity of taking $\lambda_1 \to 0$ if $K_1 \to \infty$ (eq. (22)) to obtain finite results in the local theory (cf, also ref [4])) We first note that performing the limit $K_1 \to \infty$ without passing simultaneously to $\lambda_1 \to 0$ would lead to a divergent theory These divergencies, however, do not manifest themselves in terms of infinite values of observable quantities as perturbation theory and the usual renormalization formalism would seem to indicate Instead, they just give rise to a free theory, e g , to vanishing cross sections, in accordance with the point of view maintained by Landau and others [5]) on the significance of a divergent theory We shall show this in a forthcoming paper in connection with the $Z_3 = 0$ limit of a Yukawa theory

The vanishing of the cross section can be understood by interpreting the interaction between A particles as being due to a potential of range $1/K_1$ and strength $2\lambda_1(2\pi)^3 K_1^3$ (cf, eq (38)) On account of the small range of the potential ($1/K_1 \to 0$) one expects already classicially a total cross section of the order $1/K_1^2 \to 0$ At first sight one might surmise that this result could be avoided by means of an increase of the depth of the potential, i e , by taking $|\lambda_1| \to \infty$ This, however, would not change the classical argument since classically the total cross section depends only on the dimension of the potential and not on its depth To obtain a non-zero result we must use an essentially quantal property, viz , the possibility of resonances inside the potential To this end let us introduce a propagation vector η in the interior of the potential

$$V = 2\lambda_1(2\pi)^3 \delta(\boldsymbol{x}) = 2\lambda_1(2\pi)^3 K_1^3 \tag{26}$$

(cf eq (38)), viz

$$\eta = \left(2m(E-V)\right)^{\frac{1}{2}} = \left(2m(E-1\lambda_1(2\pi)^3 K_1^3\right)^{\frac{1}{2}} \tag{27}$$

The potential has the range $1/K_1$. Requiring that precisely one wave length be contained in the potential, i e , that $2\pi/\eta \to 1/K_1$, we find immediately that $2\pi K_1/\eta \to 1$ together with (27) implies that $8\pi\lambda_1 m K_1 \to -1$ The way in which the limit $8\pi\lambda_1 m K_1 \to -1$ is reached is precisely characterized by our constant β in (22), $\beta = \lim[mK_1 + 1/8\pi\lambda_1]$ Thus we see that the non-zero results obtained in the limit (22) are due to a near-to-resonance-effect, β being a measure for the proximity to resonance For $\beta \to 0$ we have a true resonance, of course We might mention that it is this type of effect which makes a field theory with $Z_1 = 0$ different from a free one [8])

The limit (13) shows that the B particles do not manifest themselves in any real process since an energy $E \geqq 3M$ would be necessary for this to happen and $M \to \infty$ Therefore, considering only processes of finite energy, the Hamiltonian H (eq (7)) is equivalent – with respect to binding and scattering effects – to the Hamiltonian

$$H' = \int \mathrm{d}\boldsymbol{k} \left(\frac{1}{2m} \boldsymbol{k}^2 + m_0\right) a^+(\boldsymbol{k})a(\boldsymbol{k})$$

$$+\lambda_1 \int \prod_1^4 \mathrm{d}\boldsymbol{k}_i \delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)a^+(\boldsymbol{k}_1)a^+(\boldsymbol{k}_2)a(\boldsymbol{k}_3)a(\boldsymbol{k}_4) \tag{28}$$

in which the effect of the B particles is concentrated in the mass value m_0 The Hamiltonian H' is just the Hamiltonian of a direct Fermi interaction In contrast with conventional theories, however, m_0 is not an arbitrary parameter but a well-determinded quantity Similarly, the coupling of the compound-particle to its constituents turns out to be well-determined (cf sect 5)

The bound state $|2a\rangle$ for the new Hamiltonian H' is now given by

$$|2a\rangle = N \int \mathrm{d}\boldsymbol{k} f(\boldsymbol{k})a^+(\boldsymbol{k})a^+(-\boldsymbol{k})|0\rangle, \tag{29}$$

where the wave function normalization is

$$N = \left(2 \int \mathrm{d}\boldsymbol{k}|f(\boldsymbol{k})|^2\right)^{-\frac{1}{2}} \tag{30}$$

The eigenvalue E of the bound state

$$H'|2a\rangle = E|2a\rangle$$

coincides with eq (23) and $f(\boldsymbol{k})$ with the quantity given by eq (25) Similarly, the one-particle solution pertaining to H' coincides with that pertaining to H, $|a, \boldsymbol{k}\rangle = a^+(\boldsymbol{k})|0\rangle$ The equivalence between H and H' persists also in higher sectors since the limit $M \to \infty$ prevents the intervention of B particles in any real process of finite energy On the other hand, virtual B particles are always related to a term of the type $1/(E-3M)$ (cf eq (10)) Therefore, the only processes contributing with a non-zero result are those in which the largest number of intermediate integrations compensates the term $1/M$ in $1/(E-3M)$ by the ultraviolet factor K^6 These processes correspond to the emission and absorption of 3 B particles by the same A particle and thus contribute only to the self-energy of the latter Hence, no scattering or binding of A particles through intermediate B particles can occur The argument is clearly independent of the sector considered (for processes of finite energy)

It would of course have been a trivial matter to write down the Fermi interaction

H' between the massive particles from the outset But this would have given no indication of the origin of the mass m_0 The vital point is that m_0 has been generated by the interaction which simultaneously binds the particles together while they are getting massive

4. Approximation Schemes

We now confront the exact solutions with those furnished by various approximation schemes The following notation will be used $F = (2\pi)^{-4}\int d^4p \exp(ip\cdot x)$ is the Fourier operator, $\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$, $t = t_1 - t_2$, $d^4p = d\boldsymbol{p}\,dp_0$, $x = (x_\nu)$, $p\cdot x = \boldsymbol{x}\cdot\boldsymbol{p} - tp_0$, Θ the step function.

The exact B particle propagator is the free one since there is no dressing of B particles through A particles Hence,

$$S_F^b(\boldsymbol{x}, t) = \langle 0|T(B(\boldsymbol{x}_1, t_1)B^+(\boldsymbol{x}_2, t_2))|0\rangle = \delta(\boldsymbol{x})\Theta(t)\exp(-iMt), \tag{31a}$$

$$S_F^b(\boldsymbol{x}, t) = FS_F^b(\boldsymbol{p}, p_0) = F[i/(p_0 - M + i\varepsilon)]. \tag{31b}$$

The exact A particle propagator in the limit $M \to \infty$, $K_2 \to \infty$ is given by

$$\begin{aligned} S_F^a(\boldsymbol{x}, t) &= \langle 0|T(A(\boldsymbol{x}_1, t_1)A^+(\boldsymbol{x}_2, t_2))|0\rangle, \\ S_F^a(\boldsymbol{x}, t) &= FS_F^a(\boldsymbol{p}, p_0) = F[i/(p_0 - \boldsymbol{p}^2/2m - m_0 - i\varepsilon)] \end{aligned} \tag{32}$$

There are clearly no scattering states of B particles contributing (cf. the appendix). From eq (1) we have the following equation of motion for the A field

$$i\partial A/\partial t = -\Delta A/2m + 2(2\pi)^3\lambda_1 A^+AA + (2\pi)^3\lambda_2 BBB. \tag{33}$$

We note that

$$\langle 0|TA^+A^+|0\rangle = \langle 0|TB^+B^+|0\rangle = \langle 0|TA^+B|0\rangle = 0, \tag{34}$$

since $n_a + \frac{1}{3}n_b = \text{const.}$ (cf, eq (5))

4 1 PERTURBATION APPROACH

The self-energy of a single A particle is given by

$$\sum(\boldsymbol{x}, t) = i3!\lambda_2^2(2\pi)^6[S_F^b(\boldsymbol{x}, t)]^3 \tag{35}$$

Passing to momentum space and taking account of the cutoff through the insertion of $\prod_1^3\Theta(K_2^2 - p_i^2)$ into the integral, we find

$$\sum(p_\nu) = -3!\lambda_2^2 G(\boldsymbol{p})/(p_0 - 3M)$$

Taking the limit (13) gives $\sum(p_\nu) = \sum(p_0) = m_0$ and thus we find again the A particle propagator $S_F^a(p_\nu) = i/(p_0 - \boldsymbol{p}^2/m - \Sigma)$ with $\Sigma = m_0$

4 2 ONE-TIME TAMM-DANCOFF APPROXIMATION

The one-tıme Tamm-Dancloff amplıtude [6]) for the bound state $|2a\rangle$ ıs gıven by

$$\varphi(\boldsymbol{x}_1,\boldsymbol{x}_2, t) = \langle 0|T(A(\boldsymbol{x}_1, t)A(\boldsymbol{x}_2, t))|2a\rangle,$$

the T product for equal tımes beıng defined by the average for $t = \pm\varepsilon, \varepsilon \to 0$ Invarıance wıth respect to tıme translatıon ımplıes that

$$\varphi(\boldsymbol{x}_1, \boldsymbol{x}_2, t) = \varphi(\boldsymbol{x}_1, \boldsymbol{x}_2) \exp(-\iota Et),$$

where E ıs the energy of the bound state. From the equatıon of motıon ıt follows that $(\varphi = \varphi(\boldsymbol{x}_1, \boldsymbol{x}_2))$

$$\begin{aligned} E\varphi = & -\Delta_1\varphi/2m - \Delta_2\varphi/2m \\ & +2(2\pi)^3\lambda_1\langle 0|T \quad A^+(\boldsymbol{x}_1, t)A(\boldsymbol{x}_1, t)A(\boldsymbol{x}_1, t) \cdot A(\boldsymbol{x}_2, t)|2a\rangle \\ & +2(2\pi)^3\lambda_1\langle 0|TA(\boldsymbol{x}_1, t) \quad A^+(\boldsymbol{x}_2, t)A(\boldsymbol{x}_2, t)A(\boldsymbol{x}_2, t) \cdot |2a\rangle \\ & +(2\pi)^3\lambda_2\langle 0|T(B(\boldsymbol{x}_1, t))^3A(\boldsymbol{x}_2, t)|2a\rangle \\ & +(2\pi)^3\lambda_2\langle 0|TA(\boldsymbol{x}_1, t)(B(\boldsymbol{x}_2, t))^3|2a\rangle. \end{aligned}$$

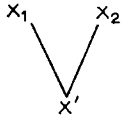

Fıg 6 Kernel for the bound state ın one-tıme Tamm-Dancoff approxımatıon, no partıcle beıng dressed

In the $n = 2$ approxımatıon we neglect normal products wıth more than two operators By Wıck's rules and ın vırtue of eq (34) we then arrıve at

$$\begin{aligned} E\varphi = & -\Delta_1\varphi/2m - \Delta_2\varphi/2m \\ & +\lambda_1(2\pi)^3\delta(\boldsymbol{x}_1-\boldsymbol{x}_2)\varphi(\boldsymbol{x}_1, \boldsymbol{x}_1)+\lambda_1(2\pi)^3\delta(\boldsymbol{x}_1-\boldsymbol{x}_2)\varphi(\boldsymbol{x}_2, \boldsymbol{x}_2), \end{aligned} \tag{36}$$

sınce $S_F^a(\boldsymbol{x}, t = 0) = \frac{1}{2}\delta(\boldsymbol{x})$ In momentum space, eq (36) reads

$$E\varphi(\boldsymbol{p}_1, \boldsymbol{p}_2) = (\boldsymbol{p}_1^2+\boldsymbol{p}_2^2)/2m+2\lambda_1\int \mathrm{d}\boldsymbol{p}'\mathrm{d}\boldsymbol{p}''\varphi(\boldsymbol{p}', \boldsymbol{p}'')\delta(\boldsymbol{p}_1+\boldsymbol{p}_2-\boldsymbol{p}'-\boldsymbol{p}''),$$

and ın the c.m s , where $\varphi(\boldsymbol{p}_1, \boldsymbol{p}_2) = f_1(\boldsymbol{p})\delta(\boldsymbol{p}_1+\boldsymbol{p}_2)$, we find

$$Ef_1(\boldsymbol{p}) = \boldsymbol{p}^2 f_1(\boldsymbol{p})/m+2\lambda_1\int \mathrm{d}\boldsymbol{p}'f(\boldsymbol{p}'), \tag{37}$$

or ın relatıve coordınates of configuratıon space

$$Ef_1(\boldsymbol{x})+\Delta f_1(\boldsymbol{x})/m-2\lambda_1(2\pi)^3\delta(\boldsymbol{x})f_1(\boldsymbol{x}) = 0. \tag{38}$$

This equation may be viewed as a Schrodinger equation with a δ potential From eq (37) it follows that

$$\int_0^{K_1} \mathrm{d}|\boldsymbol{k}|\boldsymbol{k}^2(E-\boldsymbol{k}^2/m)^{-1} = 1/8\pi\lambda_1 \tag{39}$$

In contrast with the exact equation (20), eq (39) does not possess positive energy solutions for the integrand has a pole for $E > 0$ In the limit (22) the solution of eq (39) is $E = -B$ with the binding energy B given by eq (24) The pole mentioned is due to the kinetic part of the A particle, the rest-energy of which remains zero This is a general aspect of the one-time TD approximation which in a modified form (complex energies) also appears in relativistic theories As has been mentioned in sect 1, the impossibility of obtaining positive energy solutions is the physical consequence of the fact that in the one-time TD the missing relative-time correlation does not permit the mass term m_0 to appear in the approximate eigenvalue equation, $S_F^a(\boldsymbol{x}, t_1-t_2 = 0) = \frac{1}{2}\delta(\boldsymbol{x})$ is independent of the mass Hence, dressing effects are neglected

4 3 TWO-TIME TAMM-DANCOFF APPROXIMATION

For the two-time TD amplitude

$$\phi(x_1, x_2) = \langle 0|TA(x_1)A(x_2)|2a\rangle, \qquad x_i = (\boldsymbol{x}_i, t_i), \tag{40}$$

we obtain from eq (33) the equation

$$i\partial\phi/\partial t_1 = -\Delta_1\phi/2m+2\lambda_1(2\pi)^3\langle 0|TA^+(x_1)A(x_2)|0\rangle\langle 0|TA(x_1)A(x_2)|2a\rangle, \tag{41}$$

if normal products with more than four operators are neglected With $\phi(p_1, p_2) = \int \mathrm{d}^4x_1\, \mathrm{d}^2x_2\phi \exp(-ip_1x_1-ip_2\; x_2)$, we find that

$$\phi(p_1, p_2) = -\frac{i}{\pi} S_F^a(p_1)[S_F^a(p_2)]_{m_0=0}\lambda_1\int \mathrm{d}^4p'\mathrm{d}^4p''\phi(p', p'')\delta(p_1+p_2-p'-p''),$$

and in the c m s , where $\phi(p_1, p_2) = \phi(p)\delta(\boldsymbol{p}_1+\boldsymbol{p}_2)\delta(E-p_1^0-p_2^0)$, we find for the energy E of the bound state the equation

$$\phi(p) = -\frac{i}{\pi} S_F^a(\boldsymbol{p}, -p_0+\tfrac{1}{2}E)[S_F^a(\boldsymbol{p}, p_0+\tfrac{1}{2}E)]_{m_0=0}\lambda_1\int \mathrm{d}^4p'\phi(p'), \tag{42}$$

i e ,

$$\frac{i\lambda_1}{\pi}\int \mathrm{d}^4p\, S_F^a(\boldsymbol{p}, -p_0+\tfrac{1}{2}E)[S_F^a(\boldsymbol{p}, p_0+\tfrac{1}{2}E)]_{m_0=0} = 1 \tag{43}$$

Integration over p_0 yields

$$\int_0^{K_1} \mathrm{d}|\boldsymbol{k}|\boldsymbol{k}^2(E-\boldsymbol{k}^2/m-m_0)^{-1} = 1/8\pi\lambda_1, \tag{44}$$

which, in the limit (22), gives the eigenvalue

$$E = m_0 - B, \tag{45}$$

which differs from the exact result eq (23) by the amount of just one A particle mass Looking at eq (42) we see that this is due to the fact that only one of the A particle propagators contains the mass, the other being a free one Thus the $n = 2$, two-time TD approximation takes just the dressing of one of the constituents of the bound state into account, the other one remaining massless Only higher approximations of the two-time TD method furnish the missing mass of the second particle This

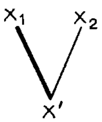

Fig 7 Kernel for the bound state in two-time Tamm-Dancoff approximation, one particle being dressed

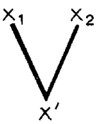

Fig 8 Kernel for the bound state in the Bethe-Salpeter equation, both particles being dresses

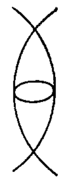

Fig 9 Higher binding graph to the Bethe-Salpeter equation

also holds for relativistic theories If already in these theories the present approximation yields results in agreement with experiment, then in higher approximations not only the mass of the second, not yet dressed particle should increase but so also should the binding energy in order to compensate for the excess Such an effect is described by the diagram in fig 9

4 4 ONE-TIME versus TWO-TIME TAMM-DANCOFF METHOD

We prove here the equivalence of the two-time TD approximation with an infinite set of one-time TD equations

Let $\tau = t_1 - t_2$ be the relative-time, $T = \frac{1}{2}(t_1 + t_2)$ the c m s time The two-time TD amplitude can be written as

$$\phi(x_1, x_2) = \phi(T+\tfrac{1}{2}\tau, T-\tfrac{1}{2}\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = \chi(T, \tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = \chi(T, \tau),$$

and from eq (41) it follows that χ satisfies the equation

$$i\partial\chi/\partial\tau + \tfrac{1}{2}i\partial\chi/\partial T = -\Delta_1 \chi/2m + 2(2\pi)^3 \lambda_1 S_F^a(\boldsymbol{x}, \tau)\chi(T+\tfrac{1}{2}\tau, 0)$$

With $\chi(T, \tau)$ $\chi(\tau)\exp = (-iET)$, we have

$$i\partial\chi(\tau)/\partial\tau + \tfrac{1}{2}E\chi(\tau) = -\Delta_1 \chi/2m + 2(2\pi)^3 \lambda_1 S_F^a(-\tau) \exp(-\tfrac{1}{2}iET)\chi(0), \tag{46}$$

where the dependence on spatial coordinates has not been explicitly written. For $\tau = 0$, setting $\chi(0) = \chi(0, \boldsymbol{x}) = f_2(\boldsymbol{x})$ and using $S_F^a(\boldsymbol{x}, \sigma) = \frac{1}{2}\delta(\boldsymbol{x})$, eq (46) yields

$$Ef_2(\boldsymbol{x}) + \Delta f_2(\boldsymbol{x})/m - 2\lambda_1(2\pi)^3\delta(\boldsymbol{x})f_2(\boldsymbol{x}) = -2if_3(\boldsymbol{x}), \tag{47}$$

where by definition

$$f_3(\boldsymbol{x}) = (\partial\chi/\partial\tau)_{\tau=0} = \tfrac{1}{2}[(\partial\chi/\partial\tau)_{\tau\to+0}+(\partial\chi/\partial\tau)_{\tau\to-0}]$$

Eq (47) differs from eq (38) by the term $-2if_3(\boldsymbol{x})$. Consistency requires that $-2if_3(\boldsymbol{x}) = m_0 f_2(\boldsymbol{x})$, in order to reproduce eq. (44) Successive differentiation of eq (47) leaves us with the following system of one-time equations

$$-2if_n(\boldsymbol{x}) = Ef_{n-1}(\boldsymbol{x})+\Delta f_{n-1}(\boldsymbol{x})/m+2(2\pi)^3\lambda_1 \sum_{v=0}^{n-1} \frac{(-1)^v(v-1)!S_v(\boldsymbol{x})}{2^{n-v}v!(n-1-v)!} \times(-\tfrac{1}{2}iE)^{n-1-v}f_2(\boldsymbol{x}), \qquad n = 3, 4, 5, \ldots, \tag{48}$$

where

$$f_n(\boldsymbol{x}) = \tfrac{1}{2}[(\partial f_n/\partial\tau)_{\tau\to+0}+(\partial f_n/\partial\tau)_{\tau\to-0}],$$
$$S_v(\boldsymbol{x}) = (\partial S_F^a(\boldsymbol{x}, \tau)/\partial\tau)_{\tau\to0} = i^v(\Delta/m-m_0)^v\delta(\boldsymbol{x})$$

This completes the proof.

4 5 BETHE-SALPETER EQUATION

The Bethe-Salpeter equation [7]) for the bound state of two A particles is given by

$$\psi(x_1, x_2) = -2i(2\pi)^3\lambda_1 \int d^4x' S_F^a(x_1-x', t_1-t')S_F^a(x_2-x', t_2-t')\psi(x', x'), \tag{49}$$

with the amplitude $\psi(x_1, x_2) = \langle 0|TA(x_1)A(x_2)|2a\rangle$, which is the same as that in eq (40). The approximation made for it just characterizes the difference between the TD and BS methods In momentum space we have from eq (49)

$$\psi(p_1, p_2) = -\frac{i\lambda_1}{\pi} S_F^a(p_1)S_F^a(p_2)\int d^4p'd^4p''\psi(p', p'')\delta(p_1+p_2-p'-p''), \tag{50}$$

where

$$\psi(p', p'') = \frac{1}{(2\pi)^8}\int d^4x_1 d^4x_2 \psi(x_1, x_2)\exp(-ip'x_1-ip''x_2)$$

In the c m s , with $\psi(p_1, p_2) = \psi(\boldsymbol{p})\delta(\boldsymbol{p}_1+\boldsymbol{p}_2)\delta(E-p_1^0-p_2^0)$ and $\boldsymbol{p} = \tfrac{1}{2}(\boldsymbol{p}_1-\boldsymbol{p}_2)$ we thus arrive at the eigenvalue equation

$$\lambda_1\Pi(E) = 1, \tag{51}$$

where

$$\Pi(E) = -\frac{i}{\pi}\int d^4p\, S_F^a(\boldsymbol{p}, p_0)S_F^a(-\boldsymbol{p}, E-p_0) \tag{52}$$

Integrating over p_0' leaves us finally with

$$\int_0^{K_1} d|\boldsymbol{k}|\boldsymbol{k}^2(E-\boldsymbol{k}^2/m-2m_0)^{-1} = 1/8\pi\lambda_1, \tag{53}$$